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Inverse Trigonometric Functions

1. **Inverse Trigonometric Function** Trigonometric functions are many-one function but we know inverse of function exists, if function is bijective. If we restrict the domain of trigonometric function, these become bijective and inverse of trigonometric functions are defined within the restricted domain.

   Let $y = f(x) = \sin x$, then its inverse is $x = \sin^{-1} y$

   NOTE $\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$ as $(\sin x)^{-1} = \frac{1}{\sin x}$ or $\sin^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$ and similarly for other trigonometric functions.

2. **Domain and Range of Inverse Trigonometric Functions**

Function	Domain	Range (Principal value branches)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$
$\cot^{-1} x$	R	$]0, \pi[$
$\operatorname{cosec}^{-1} x$	$R-(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$

3. The value of an inverse trigonometric functions which lies in its principal value branch, is called the principal value of that inverse trigonometric function.

   NOTE Whenever no branch of an inverse trigonometric function is mentioned, it means we have to consider the principal value branch of that function.

4. **Properties of Inverse Trigonometric Functions**

(a) (i) $\sin^{-1}\dfrac{1}{x}=\operatorname{cosec}^{-1}x;\ x\geq 1 \text{ or } x\leq -1$ (ii) $\cos^{-1}\dfrac{1}{x}=\sec^{-1}x;\ x\geq 1 \text{ or } x\leq -1$

(iii) $\tan^{-1}\dfrac{1}{x}=\begin{cases}\cot^{-1}x; x>0\\ -\pi+\cot^{-1}x; x<0\end{cases}$

(b) (i) $\sin^{-1}(-x)=-\sin^{-1}x; x\in[-1,1]$ (ii) $\tan^{-1}(-x)=-\tan^{-1}x;\ x\in R$

(iii) $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x; |x|\geq 1$

(c) (i) $\cos^{-1}(-x)=\pi-\cos^{-1}x; x\in[-1,1]$ (ii) $\sec^{-1}(-x)=\pi-\sec^{-1}x; |x|\geq 1$

(iii) $\cot^{-1}(-x)=\pi-\cot^{-1}x; x\in R$

(d) (i) $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}; x\in[-1,1]$ (ii) $\tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}; x\in R$

(iii) $\operatorname{cosec}^{-1}x+\sec^{-1}x=\dfrac{\pi}{2}; |x|\geq 1$

(e) (i) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

(ii) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2})$

(iii) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})$

(iv) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2})$

(f) (i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right); xy<1$

(ii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\dfrac{x-y}{1+xy}\right); xy>-1$

(g) (i) $2\tan^{-1}x=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); |x|\leq 1$ (ii) $2\tan^{-1}x=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); x\geq 0$

(iii) $2\tan^{-1}x=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right); -1<x<1$

(iv) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2}); \dfrac{-1}{\sqrt{2}}\leq x\leq\dfrac{1}{\sqrt{2}}$

(v) $2\cos^{-1}x=\cos^{-1}(2x^2-1); 0\leq x\leq 1$

(h) (i) $3\sin^{-1}x=\sin^{-1}(3x-4x^3); \dfrac{-1}{2}\leq x\leq\dfrac{1}{2}$

(ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x); \dfrac{1}{2}\leq x\leq 1$

(iii) $3\tan^{-1}x=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); \dfrac{-1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}$

(i) (i) $\sin^{-1} x = \cos^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

(ii) $\cos^{-1} x = \sin^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

5. *Following substitution is used to write inverse trigonometrical functions in simplest form.*

S. No.	For	Substitute
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\theta$
(iv)	$\sqrt{a+x}$ or $\sqrt{a-x}$	$x = a\cos\theta$ or $x = a\cos 2\theta$

6. **Remember Points**

(i) Sometimes, it may be happen, when we find out the values of x, it may be possible that, some values of x does not satisfy the given equation.

(ii) While solution of an equation, do not cancel the common factor.

Previous Years' *Examinations Questions*

1 Mark *Questions*

1. Write the principal value of $\left[\cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\left(-\dfrac{1}{2}\right)\right]$. [Delhi 2013C]

2. Write the value of $\tan^{-1}\left(\dfrac{a}{b}\right) - \tan^{-1}\left(\dfrac{a-b}{a+b}\right)$. [Delhi 2013C]

3. Write the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right)$. [HOTS; Delhi 2013]

4. Write the value of $\tan\left(2\tan^{-1}\dfrac{1}{5}\right)$. [Delhi 2013]

5. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\dfrac{\sqrt{3}}{2}\right)\right]$. [All India 2013]

6. Write the principal value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$. [All India 2013]

7. Write the value of $\cos^{-1}\left(\dfrac{1}{2}\right) - 2\sin^{-1}\left(-\dfrac{1}{2}\right)$. [Delhi 2012]

8. Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$. [All India 2012]

9. Using the principal values, write the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$. [All India 2012C]

10. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$. [Delhi 2011]

11. Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$. [HOTS; Delhi 2011]

12. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. [HOTS; Delhi 2011, 2009; All India 2009]

13. What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$. [All India 2011, 2008, 2009C]

14. What is the principal value of $\tan^{-1}(-1)$? [Foreign 2011, 2008C]

15. Using the principal values, write the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. [HOTS; All India 2011C]

16. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$. [Delhi 2011C]

17. Write the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. [Delhi 2010]

18. What is the principal value of $\sec^{-1}(-2)$? [All India 2010]

19. What is the domain of the function $\sin^{-1} x$. [Foreign 2010]

20. Using the principal values, find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$. [All India 2010C]

21. If $\tan^{-1}(\sqrt{3}) + \cot^{-1} x = \frac{\pi}{2}$, then find the value of x. [All India 2010C]

22. Write the principal value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$. [Delhi 2009]

23. Using the principal values, evaluate $\tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right)$. [Delhi 2009C]

24. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. [All India 2008C]

4 Marks *Questions*

25. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$. [Delhi 2013C]

OR

Solve for x, $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$. [Delhi 2013C, 2009; All India 2009C, 2008]

26. Find the value of the following

$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$, if $|x| < 1$, $y > 0$ and $xy < 1$. [Delhi 2013]

OR

Prove that

$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$. [Delhi 2013, All India 2011, 2008C]

27. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$.

OR

Solve the following equation

$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$. [HOTS; All India 2013]

28. Prove that

$\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$. [Delhi 2012, 2010C]

29. Prove that

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

[HOTS; Delhi 2012]

30. Prove that

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right).$$

[All India 2012, Delhi 2010C, 2009]

31. Prove that

$$\sin^{-1}\left(\frac{63}{65}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right).$$

[Foreign 2012]

32. Solve for x,

$$2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x), x\neq\frac{\pi}{2}.$$

[Foreign 2012]

33. Prove the following

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2}; x\in\left(0,\frac{\pi}{4}\right).$$

[HOTS; Delhi 2011; All India 2009]

34. Find the value of

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right).$$

[Delhi 2011]

35. Prove that

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,$$

$$-\frac{1}{\sqrt{2}}\leq x\leq 1.$$

[HOTS; All India 2011, 2010C]

36. Prove that

$$2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right).$$

[All India 2011; Delhi 2009C, 2008C]

37. Prove that

$$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

[Foreign 2011]

38. Solve following equation for x,

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, x>0.$$

[Foreign 2011C, 08C; All India 2010, 2009C]

39. Prove that

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3}.$$

[All India 2011C]

40. Solve for x,

$$\cos(2\sin^{-1}x)=\frac{1}{9}, x>0.$$

[HOTS; All India 2011C]

41. Prove that

$$2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}.$$

[Delhi 2011C]

42. Solve for x,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3},$$

$$-1<x<1.$$

[HOTS; Delhi 2011C]

43. Prove that

$$\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in(0,1).$$

[HOTS; Delhi 2010]

44. Prove that

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right).$$

[Delhi 2010]

45. Prove that

$$\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

[All India 2010]

46. Prove that

$$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]=\sqrt{\frac{1+x^2}{2+x^2}}.$$

[All India 2010]

47. Solve for x,

$$\cos^{-1}x+\sin^{-1}\left(\frac{x}{2}\right)=\frac{\pi}{6}.$$

[All India 2010C]

48. Prove that

$$2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}.$$

[All India 2010C]

49. Solve for x,

$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4},\ \sqrt{6}>x>0$$ [Delhi 2010C]

50. Solve for x,

$$\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right), x>0$$

[Delhi 2010C]

51. Solve for x,
$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$. [All India 2009]

52. Prove that

$$\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}.$$

[Delhi 2009]

53. Prove that

$$\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}.$$

[All India 2009C]

54. Prove that

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

[All India 2009C; Delhi 2008, 2008C]

55. Solve for x, $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$. [Delhi 2009C]

56. Solve for x, $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$.

[Delhi 2009C]

57. Solve for x,

$$\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x, 0<x<1.$$ [Delhi 2008C]

Step-by-Step Solutions

1. $$\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)$$

$$=\cos^{-1}\frac{\sqrt{3}}{2}+\left[\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]$$

$[\because \cos^{-1}(-x)=\pi-\cos^{-1}x]$

$$=\frac{\pi}{6}+\pi-\frac{\pi}{3}=\frac{\pi+6\pi-2\pi}{6}=\frac{5\pi}{6}$$ **(1)**

2. $$\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)=\tan^{-1}\frac{\left[\frac{a}{b}-\left(\frac{a-b}{a+b}\right)\right]}{1+\frac{a}{b}\left(\frac{a-b}{a+b}\right)}$$

$$\left[\because \tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$=\tan^{-1}\left(\frac{a^2+ab-ab+b^2}{ab+b^2+a^2-ab}\right)$$ **(1/2)**

$$=\tan^{-1}\left(\frac{a^2+b^2}{a^2+b^2}\right)=\tan^{-1}1$$

$$=\tan^{-1}\frac{\pi}{4}=\frac{\pi}{4}$$ **(1/2)**

3.

> Firstly, we check the given angle is in principal value. If it is not so, then convert it.
> After that, use the identity
> $\tan^{-1}(\tan\theta)=\theta,\ \cos^{-1}(\cos\theta)=\theta$

Given that, $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(-\cos\frac{\pi}{3}\right)$$ **(1/2)**

$$=\frac{\pi}{4}+\cos^{-1}\left[\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

[$\because$ Principal value of $\cos^{-1}$ is $[0,\pi]$, so we convert in $-\cos\theta=\cos(\pi-\theta)$]

$$=\frac{\pi}{4}+\cos^{-1}\left[\cos\frac{2\pi}{3}\right]$$

$$=\frac{\pi}{4}+\frac{2\pi}{3}$$

$$=\frac{3\pi+8\pi}{12}$$

$$=\frac{11\pi}{12}$$ $[\because \cos^{-1}(\cos\theta)=\theta]$ **(1/2)**

4. Given that,

$$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left[\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)\right] \quad \textbf{(1/2)}$$

$$\left[\because 2\tan^{-1}\theta = \tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{2\times 5}{24}\right)\right] = \tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right] = \frac{5}{12}$$

$$[\because \tan(\tan^{-1}\theta) = \theta] \quad \textbf{(1/2)}$$

5. $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(2\cdot\frac{3}{4}-1\right)\right\}\right]$$

$$[\because 2\cos^{-1}x = \cos^{-1}(2x^2-1)]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(\frac{3}{2}-1\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(\frac{1}{2}\right)\right\}\right] \quad \textbf{(1/2)}$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\cdot\cos\frac{\pi}{3}\right\}\right]$$

$$= \tan^{-1}\left[2\sin\frac{\pi}{3}\right] = \tan^{-1}\left(2\cdot\frac{\sqrt{3}}{2}\right)$$

$$[\because \cos^{-1}(\cos\theta) = \theta]$$

$$= \tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$$[\because \tan^{-1}(\tan\theta) = \theta] \quad \textbf{(1/2)}$$

6. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\}$$

[$\because$ Principal value of $\cot^{-1}$ is] 0, π [

$\therefore \cot^{-1}(-x) = \pi - \cot^{-1}x$]

$$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$$

$$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right] \textbf{(1)}$$

7. Firstly, we check the given angle is in principal value. If it is so, then use the identity $\sin^{-1}(\sin\theta) = \theta$ and $\cos(\cos^{-1}\theta) = \theta$, otherwise to convert in principal value.

$$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \cos^{-1}\left(\cos\frac{\pi}{3}\right) - 2\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

[$\because$ Principal value for $\cos^{-1}x$ is $(0, \pi)$ and that of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$]

$$= \frac{\pi}{3} - 2\left(-\frac{\pi}{6}\right)$$

$$[\because \cos^{-1}(\cos\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta]$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3} \quad \textbf{(1)}$$

8. Given expression is not standard identity, so we separately find the value of $\tan^{-1}(\sqrt{3})$ and $\sec^{-1}(-2)$, then simplify it.

We know that, the principal value for $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and that of $\sec^{-1}x$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

So, $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \sec\frac{2\pi}{3} = -2\right]$$

$$= \frac{\pi}{3} - \frac{2\pi}{3}$$

$$= \frac{-\pi}{3}$$

$$[\because \tan^{-1}(\tan\theta) = \theta \text{ and } \sec^{-1}(\sec\theta) = \theta] \textbf{(1)}$$

9. We know that, the principal value for $\cos^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{3}$ as $\cos^{-1}x \in [0, \pi]$ and $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$ as $\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \cos^{-1}\left(\cos\frac{\pi}{3}\right) + 2\sin^{-1}\left(\sin\frac{\pi}{6}\right)$$

$$\left[\because \cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$[\because \cos^{-1}(\cos\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta] \textbf{ (1)}$$

10. We are given that

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$[\because \sin^{-1}(-\theta)=-\sin^{-1}\theta]$

$$=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]\quad\left[\because \sin\frac{\pi}{6}=\frac{1}{2}\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]=\sin\frac{\pi}{2}=1$$

$[\because \sin^{-1}(\sin\theta)=\theta]$ **(1)**

NOTE Please be careful that we do not write $\sin^{-1}(-\sin\theta)=\theta$

11. Firstly, we check the given angle is in principal value. If it is so, then use the identity $\tan^{-1}(\tan\theta)=\theta$.

Principal value of $\tan^{-1}\theta$ is $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$\therefore$ The principal value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right]$$

Here, $\frac{3\pi}{4}\notin\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

So, we write $\frac{3\pi}{4}$ as $\pi-\frac{\pi}{4}$

$$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)\quad[\because \tan(\pi-\theta)=-\tan\theta]$$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]\quad[\because -\tan\theta=\tan(-\theta)]$$

$$=-\frac{\pi}{4}\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$$

$$\therefore\quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=-\frac{\pi}{4}$$ **(1)**

NOTE Please be careful, we do not write $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\frac{3\pi}{4}$ because $\frac{3\pi}{4}\notin\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

12. Firstly, we reduce the quantity $\frac{7\pi}{6}$ according to the interval $[0,\pi]$ i.e., by using $\cos\theta=\cos(2\pi-\theta)$ and then use the identity $\cos^{-1}(\cos\theta)=\theta$.

We know that, the principal value of $\cos^{-1}\theta$ is $[0,\pi]$.

$$\therefore\quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi-\frac{5\pi}{6}\right)\right]$$

$\left[\because \frac{7\pi}{6}\notin[0,\pi]\right.$

$\left.\therefore \text{ we can write as } \frac{2\pi}{3}=\pi-\frac{\pi}{3}\right]$

$$=\cos^{-1}\left[\cos\frac{5\pi}{6}\right]\quad\left[\because \cos(2\pi-\theta)=\cos\theta \text{ and } \frac{5\pi}{6}\in[0,\pi]\right]$$

$$\therefore\quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\frac{5\pi}{6}$$ **(1)**

NOTE Please be careful, we do not write $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\frac{7\pi}{6}$ as $\frac{7\pi}{6}\notin[0,\pi]$.

13. As the principal value of $\cos^{-1}\theta$ is $[0,\pi]$ and for $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$\therefore\quad \cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$=\frac{2\pi}{3}+\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$$

$\left[\because \frac{2\pi}{3}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right.$

$\left.\therefore \text{ we can write } \frac{2\pi}{3} \text{ as } \left(\pi-\frac{\pi}{3}\right)\right]$

$$=\frac{2\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)\quad[\because \sin(\pi-\theta)=\sin\theta]$$

$$=\frac{2\pi}{3}+\frac{\pi}{3}$$

$$=\frac{3\pi}{3}=\pi\quad[\because \sin^{-1}(\sin\theta)=\theta]$$ **(1)**

14. We know that, the principal value of $\tan^{-1}(\theta)$ is $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

$$\therefore\quad \tan^{-1}(-1)=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)\quad\left[\because \tan\frac{\pi}{4}=1\right]$$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]\quad[\because -\tan\theta=\tan(-\theta)]$$

$$=-\frac{\pi}{4}\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$$

$$\therefore\quad \tan^{-1}(-1)=-\frac{\pi}{4}$$ **(1)**

15. We know that, the principal value of $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{3}\right)$

$= \sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right]$ $\left[\because \sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} \text{ as } \sin(-\theta) = -\sin\theta\right]$

$= -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $[\because \sin^{-1}(\sin\theta) = \theta]$

$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ **(1)**

NOTE Please be careful that, we do not write $\sin^{-1}(-\sin\theta) = \theta$ because firstly we write this '–' sign in angle side and then use property.

16. We know that, the principal value of $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$

$= \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$ $\left[\because \sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} \text{ as } \sin(-\theta) = -\sin\theta\right]$

$= -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $[\because \sin^{-1}(\sin\theta) = \theta]$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ **(1)**

NOTE Principal value of any inverse function is unique.

17. As principal value of $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right)$ $\left[\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$

$= \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$[\because \sin^{-1}(\sin\theta) = \theta]$

$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ **(1)**

18. We know that, the principal value of $\sec^{-1}\theta$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2) = \sec^{-1}\left(-\sec\frac{\pi}{3}\right) \neq \frac{-\pi}{3}$

as $\frac{-\pi}{3} \notin [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Now, $\sec^{-1}(-2) = \sec^{-1}\left[\sec\left(\pi - \frac{\pi}{3}\right)\right]$

$[\because \sec(\pi - \theta) = -\sec\theta]$

$= \sec^{-1}\left(\sec\frac{2\pi}{3}\right) = \frac{2\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$[\because \sec^{-1}(\sec\theta) = \theta]$

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$ **(1)**

19. The domain of $\sin^{-1}x$ is $-1 \le x \le 1$. **(1)**

20. As the principal value of $\cos^{-1}\theta$ is $[0, \pi]$.

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) \neq \frac{13\pi}{6}$ as $\frac{13\pi}{6} \notin [0, \pi]$

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$= \cos^{-1}\left(\cos\frac{\pi}{6}\right)$ $[\because \cos(2\pi + \theta) = \cos\theta]$

$= \frac{\pi}{6} \in [0, \pi]$ $[\because \cos^{-1}(\cos\theta) = \theta]$

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$ **(1)**

21. Given that, $\tan^{-1}\sqrt{3} + \cot^{-1}x = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{3} = \frac{\pi}{2} - \cot^{-1}x$

$\Rightarrow \tan^{-1}\sqrt{3} = \tan^{-1}x$ $\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

On equating, we get

$x = \sqrt{3}$ **(1)**

22. As we know that, the principal value of $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right] \neq \frac{3\pi}{5}$ $\left[\because \frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

Now, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{5}\right)\right]$

$= \sin^{-1}\left(\sin\frac{2\pi}{5}\right)$ [$\because$ $\sin(\pi - \theta) = \sin\theta$]

$= \frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ [$\because \sin^{-1}(\sin\theta) = \theta$]

$\therefore$ $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$ **(1)**

23. Principal value of $\tan^{-1}\theta$ is $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and that of $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore$ $\tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}\right]$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$

$\left[\begin{array}{l}\because \sin(-\theta) = -\sin\theta \\ \therefore \sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}\end{array}\right]$

$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

[$\because \tan^{-1}(\tan\theta) = \theta$ and $\sin^{-1}(\sin\theta) = \theta$] **(1)**

24. $\cos^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\left(\cos\frac{\pi}{6}\right)$ $\left[\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$

Since, $\frac{\pi}{6} \in [0, \pi]$

{As principal value of $\cos^{-1}\theta$ is $[0, \pi]$}

$\therefore$ $\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$ **(1)**

25. To prove, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

LHS $= \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

$= \sin^{-1}\left[\frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}\right]$ **(1)**

[$\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}$]

$= \sin^{-1}\left(\frac{8}{17}\sqrt{1-\frac{9}{25}} + \frac{3}{5}\sqrt{1-\frac{64}{289}}\right)$

$= \sin^{-1}\left(\frac{8}{17}\sqrt{\frac{16}{25}} + \frac{3}{5}\sqrt{\frac{225}{289}}\right)$ **(1)**

$= \sin^{-1}\left(\frac{8}{17} \times \frac{4}{5} + \frac{3}{5} \times \frac{15}{17}\right) = \sin^{-1}\left(\frac{32}{85} + \frac{45}{85}\right)$

$= \sin^{-1}\left(\frac{77}{85}\right) = \tan^{-1}\left[\frac{77/85}{\sqrt{1-(77/85)^2}}\right]$ **(1)**

$\left[\because \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}\right]$

$= \tan^{-1}\left(\frac{77/85}{\sqrt{1-5929/7225}}\right)$

$= \tan^{-1}\left(\frac{77/85}{\sqrt{1296/7225}}\right) = \tan^{-1}\left(\frac{77/85}{36/85}\right)$

$= \tan^{-1}\left(\frac{77}{36}\right)$ **(1)**

OR

Given equation is $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\left(\frac{3x+2x}{1-3x \times 2x}\right) = \frac{\pi}{4}$ **(1)**

$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$

$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4}$

[$\because \tan^{-1}(\theta) = \phi \Rightarrow \theta = \tan\phi$]

$\Rightarrow$ $\frac{5x}{1-6x^2} = 1$

$\Rightarrow$ $5x = 1 - 6x^2$ **(1)**

$\Rightarrow$ $6x^2 + 5x - 1 = 0$

$\Rightarrow$ $6x^2 + 6x - x - 1 = 0$

$\Rightarrow$ $6x(x+1) - 1(x+1) = 0$ **(1)**

$\Rightarrow$ $(6x-1)(x+1) = 0$

$\Rightarrow$ $6x - 1 = 0$ or $x + 1 = 0$

$\Rightarrow$ $x = \frac{1}{6}$ or $x = -1$ **(1)**

But $x = -1$ does not satisfy the given equation.

Hence, value of x is $\frac{1}{6}$.

26. Firstly, we use the relation
$2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ to convert into $\tan^{-1}x$. So, we use identity relation $\tan(\tan^{-1}\theta)=\theta$.

$$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

$$=\tan\left[\frac{1}{2}(2\tan^{-1}x)+\frac{1}{2}(2\tan^{-1}y)\right] \quad \textbf{(2)}$$

$$\left[\because 2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$=\tan(\tan^{-1}x+\tan^{-1}y)$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy} \qquad [\because \tan(\tan^{-1}\theta)=\theta] \quad \textbf{(2)}$$

OR

To prove,

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$$

$$\therefore \quad \text{LHS}=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{10}}\right)+\tan^{-1}\left(\frac{1}{8}\right) \quad \textbf{(1½)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1\right]$$

$$=\tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{72}}\right) \quad \textbf{(1½)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1\right]$$

$$=\tan^{-1}\left(\frac{56+9}{72-7}\right)=\tan^{-1}\left(\frac{65}{65}\right)$$

$$=\tan^{-1}(1)=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}=\text{RHS} \quad \textbf{(1)}$$

Hence proved.

27. To prove, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

$$\text{LHS}=\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \quad \ldots\text{(i)}$$

Let $\quad \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\theta \quad \ldots\text{(ii)}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{3}{4}\right)=2\theta \Rightarrow \sin 2\theta=\frac{3}{4} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{2\tan\theta}{1+\tan^2\theta}=\frac{3}{4}$$

$$\left[\because \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow \quad 8\tan\theta=3+3\tan^2\theta$$

$$\Rightarrow 3\tan^2\theta-8\tan\theta+3=0$$

Now, by Sridharacharya's rule

$$\tan\theta=\frac{8\pm\sqrt{64-36}}{2\times 3}=\frac{8\pm\sqrt{28}}{6} \quad \textbf{(1)}$$

$$\Rightarrow \tan\theta=\frac{8\pm 2\sqrt{7}}{6}=\frac{4\pm\sqrt{7}}{3}$$

$$\Rightarrow \quad \theta=\tan^{-1}\left(\frac{4\pm\sqrt{7}}{3}\right)$$

$$[\because \tan\theta=\phi \Rightarrow \theta=\tan^{-1}\phi]$$

$$\Rightarrow \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\tan^{-1}\left(\frac{4\pm\sqrt{7}}{3}\right)$$

[From Eq. (ii)] **(1)**

Taking (–)ve sign,

$$\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)=\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)$$

On taking tan both sides, we get

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\tan\left\{\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)\right\}$$

$\Rightarrow \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

$[\because \tan(\tan^{-1}\theta) = \theta]$ **(1)**

$\therefore$ LHS = RHS

Hence proved.

OR

Given that, $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$\Rightarrow \sin\left\{\frac{\pi}{2} - \tan^{-1}x\right\} = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$\left[\because \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\right]$ **(1)**

On equating both sides, we get

$\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}\frac{3}{4}$ **(1)**

$\Rightarrow \tan^{-1}x + \cot^{-1}\frac{3}{4} = \frac{\pi}{2}$ **(1)**

This is only possible when $x = \frac{3}{4}$.

$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in R\right]$ **(1)**

28. To prove, $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$

Let $\sin^{-1}\left(\frac{8}{17}\right) = x$ ...(i)

and $\sin^{-1}\left(\frac{3}{5}\right) = y$ ...(ii)

$\Rightarrow \sin x = \frac{8}{17}$ and $\sin y = \frac{3}{5}$ **(1)**

$\because \cos^2 x = 1 - \sin^2 x$

$= 1 - \frac{64}{289} = \frac{225}{289}$

$\Rightarrow \cos x = \sqrt{\frac{225}{289}}$

$\Rightarrow \cos x = \frac{15}{17}$ **(1)**

Also, $\cos^2 y = 1 - \sin^2 y = 1 - \frac{9}{25}$

$\Rightarrow \cos y = \sqrt{\frac{16}{25}}$

$\Rightarrow \cos y = \frac{4}{5}$ **(1)**

Now, we know that,

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\Rightarrow \cos(x+y) = \left(\frac{15}{17}\times\frac{4}{5}\right) - \left(\frac{8}{17}\times\frac{3}{5}\right)$

$\Rightarrow \cos(x+y) = \frac{60}{85} - \frac{24}{85} = \frac{36}{85}$

$\Rightarrow x + y = \cos^{-1}\left(\frac{36}{85}\right)$

$\Rightarrow \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$ **(1)**

[$\because$ From Eqs. (i) and (ii)]

Hence proved.

29. Firstly, use the relation $\cos\theta = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and after that use the relation $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ and simplify it.

$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$

$\left[\because \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$ **(1)**

$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right]$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$= \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right)$ **(1)**

On dividing the numerator and denominator by $\cos\frac{x}{2}$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{x}{2}}\right) \quad \textbf{(1)}$$

$$\left[\because 1 = \tan\frac{\pi}{4} \text{ and } 1\cdot\tan\frac{x}{2} = \tan\frac{\pi}{4}\times\tan\frac{x}{2}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$\left[\because \frac{\tan A - \tan B}{1+\tan A\tan B} = \tan(A-B)\right]$$

$$= \frac{\pi}{4} - \frac{x}{2} \quad [\because \tan^{-1}(\tan\theta) = \theta]$$

$$\left[\because x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$= \text{RHS}$ **(1)**

$\therefore$ LHS = RHS

Hence proved.

30. To prove,

$$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

Let $\cos^{-1}\left(\frac{4}{5}\right) = x$ ...(i)

and $\cos^{-1}\left(\frac{12}{13}\right) = y$ ...(ii)

$\Rightarrow \cos x = \frac{4}{5}$ and $\cos y = \frac{12}{13}$ **(1)**

We know that,

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\therefore \quad \sin x = \sqrt{\frac{9}{25}} \Rightarrow \sin x = \frac{3}{5}$$

and $$\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$$

$$\therefore \quad \sin y = \sqrt{\frac{25}{169}} \Rightarrow \sin y = \frac{5}{13} \quad \textbf{(1)}$$

Now, we know that,

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\Rightarrow \cos(x+y) = \left(\frac{4}{5}\times\frac{12}{13}\right) - \left(\frac{3}{5}\times\frac{5}{13}\right) = \frac{48}{65} - \frac{15}{65}$$

$$\Rightarrow \cos(x+y) = \frac{33}{65} \quad \textbf{(1)}$$

$$\Rightarrow x + y = \cos^{-1}\frac{33}{65}$$

$$\Rightarrow \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$$

$$\left[\text{From Eqs. (i) and (ii), } x = \cos^{-1}\frac{4}{5} \text{ and } y = \cos^{-1}\frac{12}{13}\right] \quad \textbf{(1)}$$

$\therefore$ LHS = RHS

Hence proved.

31. To prove,

$$\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

$$\text{RHS} = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Let $\sin^{-1}\frac{5}{13} = x \Rightarrow \sin x = \frac{5}{13}$ ...(i)

and $\cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$ ...(ii) **(1)**

Also, $\cos x = \sqrt{1-\sin^2 x}$

$$= \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

and $\sin y = \sqrt{1-\cos^2 y}$

$$= \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad \textbf{(1)}$$

We know that,

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$= \frac{5}{13}\times\frac{3}{5} + \frac{12}{13}\times\frac{4}{5}$$

$$= \frac{15}{65} + \frac{48}{65} = \frac{63}{65} \quad \textbf{(1)}$$

$$\Rightarrow x + y = \sin^{-1}\left(\frac{63}{65}\right)$$

$$[\because \sin\theta = \phi \Rightarrow \theta = \sin^{-1}\phi]$$

$\Rightarrow \sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{63}{65}\right)$ **(1)**

[$\because$ From Eqs. (i) and (ii)]

$\therefore$ LHS = RHS

Hence proved.

32. To solve, $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right)=\tan^{-1}\left(\frac{2}{\cos x}\right) \quad \textbf{(1½)}$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right) \text{ and } \sec x=\frac{1}{\cos x}\right]$$

On comparing, we get

$$\Rightarrow \frac{2\sin x}{\cos^2 x}=\frac{2}{\cos x} \quad \textbf{(1)}$$

$$\Rightarrow \tan x = 1 \quad \textbf{(1/2)}$$

$$\therefore x=\tan^{-1}(1)=\frac{\pi}{4} \quad \textbf{(1)}$$

33. Using the relation

$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$

$=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$

For $x\in\left(0,\frac{\pi}{4}\right)$,

$1-\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}$

$=\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$

$$\text{LHS}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right] \quad \textbf{(1½)}$$

$$=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right) \quad \textbf{(1)}$$

$$=\cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right) \quad \textbf{(1/2)}$$

$\because$ The principal value of $\cot^{-1}x$ is $]0,\pi[$.

$$\therefore \cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}$$

$$\left[\because x\in\left(0,\frac{\pi}{4}\right)\Rightarrow\frac{x}{2}\in\left(0,\frac{\pi}{8}\right)\right] \quad \textbf{(1)}$$

$\therefore$ LHS = RHS

Hence proved.

NOTE If $x\in\left(0,\frac{\pi}{2}\right)$, then $\sqrt{1-\sin x}=\cos\frac{x}{2}-\sin\frac{x}{2}$

and if $x\in\left(\frac{\pi}{2},\pi\right)$, then $\sqrt{1-\sin x}=\sin\frac{x}{2}-\cos\frac{x}{2}$

Alternate Method

$$\text{LHS}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$$

[By rationalising denominator] **(1)**

$$=\cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}\right]$$

[$\because (a+b)(a-b)=a^2-b^2$]

$$=\cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

[$\because (a+b)^2=a^2+b^2+2ab$]

$$=\cot^{-1}\left(\frac{2+2\cos x}{2\sin x}\right)$$

[$\because \cos x=\sqrt{1-\sin^2 x}$] **(1)**

$$=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)=\cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \quad \textbf{(1)}$$

$$\left[\because 1+\cos\theta=2\cos^2\frac{\theta}{2} \text{ and } \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2} = \text{RHS}$$

$[\because \cot^{-1}(\cot\theta) = \theta]$ **(1)**

$\therefore$ LHS = RHS **Hence proved.**

34. Using the relation $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

We have, $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$= \tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right) \quad \textbf{(1½)}$$

$$\left[\because \tan^{-1}\theta - \tan^{-1}\phi = \tan^{-1}\left(\frac{\theta-\phi}{1+\theta\cdot\phi}\right)\right]$$

$$= \tan^{-1}\left[\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right) \quad \textbf{(1½)}$$

$\because$ The principal value of $\tan^{-1}x$ is $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$$\therefore \quad \tan^{-1}\left(\frac{x^2+y^2}{y^2+x^2}\right) = \tan^{-1}(1) = \frac{\pi}{4} \quad \textbf{(1)}$$

35. To prove

$$\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$$

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}\right)$$

[Put $x = \cos 2\theta$]

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}\right) \quad \textbf{(1)}$$

$$\left[\begin{array}{l}\because 1+\cos 2A = 2\cos^2 A \\ \quad 1-\cos 2A = 2\sin^2 A\end{array}\right]$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right) \quad \textbf{(1)}$$

On dividing numerator and denominator by $\cos\theta$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{\frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\theta\cdot\tan\frac{\pi}{4}}\right) \quad \textbf{(1)}$$

$$\left[\because 1 = \tan\frac{\pi}{4} \text{ and } 1\cdot\tan\theta = \tan\frac{\pi}{4}\cdot\tan\theta\right]$$

$\because$ Principal value of $\tan^{-1}x$ is $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$$= \tan^{-1}\tan\left[\left(\frac{\pi}{4} - \theta\right)\right]$$

$$\left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\cdot\tan B}\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x \quad \left[\because \theta = \frac{1}{2}\cos^{-1}x\right]$$

= RHS **(1)**

$\therefore$ LHS = RHS **Hence proved.**

NOTE We can also substitute $x = \cos\theta$, then use the relation $1+\cos\theta = 2\cos^2\frac{\theta}{2}$ and $1-\cos\theta = 2\sin^2\frac{\theta}{2}$, to remove root sign.

36. Use the relation, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2\times(1/2)}{1-(1/2)^2}\right] + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] \quad \textbf{(1½)}$$

$$= \tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{1}{3/4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right) \quad \textbf{(1½)}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right) = \tan^{-1}\frac{31}{17} = \text{RHS} \quad \textbf{(1)}$$

$\therefore$ LHS = RHS

Hence proved.

37. ***Method* I**

Let $\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$

Then, we get

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}$$

Now, $\cos^2 x = 1 - \sin^2 x = 1 - \frac{1}{9} = \frac{8}{9}$

$$\therefore \quad \cos x = \sqrt{\frac{8}{9}} \Rightarrow \cos x = \frac{2\sqrt{2}}{3} \quad \textbf{(1)}$$

Similarly, $\cos^2 y = 1 - \sin^2 y = 1 - \frac{8}{9} = \frac{1}{9}$

$$\therefore \quad \cos y = \sqrt{\frac{1}{9}} \Rightarrow \cos y = \frac{1}{3} \quad \textbf{(1)}$$

Now, $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$= \frac{1}{3}\times\frac{1}{3} + \frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}$$

$$= \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1 \quad \textbf{(1)}$$

$$\Rightarrow \quad \sin(x+y) = 1 = \sin\frac{\pi}{2}$$

$\because$ Principal value of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \quad x + y = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$$

$$\left[\because x = \sin^{-1}\left(\frac{1}{3}\right) \text{ and } y = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$\Rightarrow \quad \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$$

[Multiplying on both sides by 9/4]

$$\therefore \quad \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \quad \textbf{(1)}$$

Hence proved.

***Method* II**

To prove that

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\text{LHS} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right] \textbf{(1)}$$

$$= \frac{9}{4}\left[\cos^{-1}\left(\frac{1}{3}\right)\right] \quad \left[\because \cos^{-1}\theta = \frac{\pi}{2} - \sin^{-1}\theta\right] \quad \textbf{(1)}$$

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{1-\frac{1}{9}}\right)$$

$$[\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}] \quad \textbf{(1)}$$

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{\frac{8}{9}}\right) \quad \textbf{(1)}$$

$\therefore$ LHS = RHS

Hence proved.

38. Given equation is

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, \; x > 0$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right] = \tan^{-1}x \quad \textbf{(1½)}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$\Rightarrow \quad \tan^{-1}\left[\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right] = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\left(\frac{2-2x^2}{4x}\right) = \tan^{-1} x$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \quad \frac{1-x^2}{2x} = x \quad \Rightarrow \quad 1 - x^2 = 2x^2$

$\Rightarrow \quad 3x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{3}$

$\Rightarrow \quad x = \pm\frac{1}{\sqrt{3}}$ **(1½)**

But given, $x > 0$

$\therefore \quad x = \frac{1}{\sqrt{3}}$ **(1)**

39. To prove,

$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \left(\frac{1}{2}\right)\tan^{-1}\left(\frac{4}{3}\right)$ ...(i)

Above equation may be written as

$2\left[\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)\right] = \tan^{-1}\left(\frac{4}{3}\right)$ ...(ii)

(1/2)

Now, we prove Eq. (ii) as it is equivalent to Eq. (i).

$\text{LHS} = 2\left[\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)\right]$

$= 2\left[\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)\right]$ **(1)**

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= 2\tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right) = 2\tan^{-1}\left(\frac{17}{34}\right)$

$= 2\tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left[\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right]$ **(1)**

$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

$= \tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right) = \tan^{-1}\left(\frac{4}{3}\right) = \text{RHS}$ **(1½)**

$\therefore \quad \text{LHS} = \text{RHS}$ **Hence proved.**

40. Given equation is $\cos(2\sin^{-1} x) = \frac{1}{9}, x > 0$ ...(i)

We put $\sin^{-1} x = y$

$\Rightarrow \quad x = \sin y$ **(1/2)**

$\therefore$ Eq. (i) becomes, $\cos 2y = \frac{1}{9}$ $[\because \sin y = x]$

$\Rightarrow 1 - 2\sin^2 y = \frac{1}{9}$ $[\because \cos 2\theta = 1 - 2\sin^2\theta]$ **(1)**

$\Rightarrow \quad 2\sin^2 y = 1 - \frac{1}{9} = \frac{8}{9}$ **(1/2)**

$\Rightarrow \quad \sin^2 y = \frac{4}{9} \Rightarrow x^2 = \frac{4}{9}$ $[\because \sin y = x]$

$\therefore \quad x = \pm\frac{2}{3}$ [Taking square root] **(1)**

But given that, $x > 0$

$\therefore \quad x = \frac{2}{3}$ **(1)**

Alternate Method

Given equation is

$\cos(2\sin^{-1} x) = \frac{1}{9}, x > 0$

$\Rightarrow \quad \cos(\sin^{-1} 2x\sqrt{1-x^2}) = \frac{1}{9}$

$\left[\because 2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})\right]$ **(1)**

$\Rightarrow \cos\left[\cos^{-1}\sqrt{1-(2x\sqrt{1-x^2})^2}\right] = \frac{1}{9}$

$[\because \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}]$ **(1)**

$\Rightarrow \quad \sqrt{1-4x^2(1-x^2)} = \frac{1}{9}$ $[\because \cos(\cos^{-1}\theta) = \theta]$

On squaring both sides, we get

$81(1 - 4x^2 + 4x^4) = 1$

$\Rightarrow \quad 324x^4 - 324x^2 + 80 = 0$

$\Rightarrow \quad 81x^4 - 81x^2 + 20 = 0$

[divide both sides by 4]

$\Rightarrow \quad 81x^4 - 45x^2 - 36x^2 + 20 = 0$

$\Rightarrow \quad 9x^2(9x^2 - 5) - 4(9x^2 - 5) = 0$

$\Rightarrow \quad (9x^2 - 5)(9x^2 - 4) = 0$

$\Rightarrow \quad x^2 = \frac{5}{9} \text{ or } \frac{4}{9}$

$\Rightarrow \quad x = \pm\frac{\sqrt{5}}{3} \text{ or } \pm\frac{2}{3}$

But $\quad x > 0$

$\therefore \quad x = +\frac{\sqrt{5}}{3} \text{ or } \frac{2}{3}$ **(1)**

But here, $x = \frac{\sqrt{5}}{3}$ do not satisfy the given equation.

$\therefore \quad x = \frac{2}{3}$ is the only solution. **(1)**

NOTE While solving an equation, please be careful on squaring the equation. Sometimes, it may occur extra value, which do not satisfy the given equation.

41. Firstly, apply $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ to evaluate $2\tan^{-1}\left(\frac{3}{4}\right)$ and then apply $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ and get the desired result.

To prove that

$$2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$$

$$\text{LHS} = 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right) \quad \textbf{(1)}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{3/2}{7/16}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right) \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{24\times31-17\times7}{7\times31+24\times17}\right)$$

$$= \tan^{-1}\left(\frac{744-119}{217+408}\right) = \tan^{-1}\left(\frac{625}{625}\right)$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) \quad \left[\because 1 = \tan\frac{\pi}{4}\right]$$

$\because$ The principal value of $\tan^{-1}x$ is $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$\therefore \quad \text{LHS} = \frac{\pi}{4} = \text{RHS}$ **(2)**

$\therefore \quad \text{LHS} = \text{RHS}$

Hence proved.

42. Firstly, write $\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ by applying formula $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ and then proceed further.

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}, -1 < x < 1$$

We know that, $\cot^{-1}x = \tan^{-1}\frac{1}{x}$, so by using this result, we may write

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \textbf{(1/2)}$$

$\therefore$ The given equation becomes

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \quad \textbf{(1/2)}$$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$ **(1/2)**

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$

$\Rightarrow \quad \frac{2x}{1-x^2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad 2\sqrt{3}x = 1 - x^2$ **(1)**

$\Rightarrow \quad x^2 + 2\sqrt{3}x - 1 = 0$

$\Rightarrow \quad x = \dfrac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$

[$\because$ We know that, $x = \dfrac{-b \pm \sqrt{D}}{2a}$ where, $D = b^2 - 4ac$]

$\therefore \quad x = \dfrac{-2\sqrt{3} \pm 4}{2} = \dfrac{4 - 2\sqrt{3}}{2}$ or $\dfrac{-4 - 2\sqrt{3}}{2}$ **(1)**

$\Rightarrow x = 2 - \sqrt{3}$ or $-(2+\sqrt{3})$

But given that $-1 < x < 1$, so $x = -(2+\sqrt{3})$ is rejected.

Hence, $x = 2 - \sqrt{3}$ **(1/2)**

43. Put $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$ and then use $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

To prove, $\tan^{-1}\sqrt{x} = \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x}{1+x}\right), x \in (0, 1)$

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right] \quad \textbf{(1)}$$

On substituting $\sqrt{x} = \tan\theta$, we get

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \quad \textbf{(1)}$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) \quad \textbf{(1)}$$

$\because$ The principal value of $\cos^{-1}x$ is $[0, \pi]$.

$$\left[\because \frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A\right]$$

$= \dfrac{1}{2}(2\theta) = \theta \quad [\because \cos^{-1}(\cos\theta) = \theta]$

$= \tan^{-1}\sqrt{x} \quad [\because \theta = \tan^{-1}\sqrt{x}]$ **(1)**

$= \text{LHS}$

$\therefore \quad \text{RHS} = \text{LHS}$

Hence proved.

Alternate Method

To prove, $\tan^{-1}\sqrt{x} = \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x}{1+x}\right), x \in (0,1)$

$\text{LHS} = \tan^{-1}\sqrt{x} = \dfrac{1}{2}(2\tan^{-1}\sqrt{x})$

$$= \frac{1}{2} \times \cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right] \quad \textbf{(2)}$$

$$\left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \text{RHS} \quad \textbf{(2)}$$

$\therefore \text{LHS} = \text{RHS}$ **Hence proved.**

44. Let $\cos^{-1}\dfrac{12}{13} = x$ and $\sin^{-1}\dfrac{3}{5} = y$

So, $\cos x = \dfrac{12}{13}$ and $\sin y = \dfrac{3}{5}$ **(1)**

$\therefore \sin^2 x = 1 - \cos^2 x = 1 - \left(\dfrac{12}{13}\right)^2 = 1 - \dfrac{144}{169}$

$= \dfrac{25}{169} \quad [\because \sin^2\theta + \cos^2\theta = 1]$

and

$\cos^2 y = 1 - \sin^2 y = 1 - \left(\dfrac{3}{5}\right)^2 = 1 - \dfrac{9}{25} = \dfrac{16}{25}$

$\Rightarrow \sin x = \sqrt{\dfrac{25}{169}} = \dfrac{5}{13}$ and $\cos y = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$ **(1)**

Now, $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\Rightarrow \sin(x+y) = \left(\dfrac{5}{13} \times \dfrac{4}{5}\right) + \left(\dfrac{12}{13} \times \dfrac{3}{5}\right)$

$= \dfrac{20}{65} + \dfrac{36}{65} = \dfrac{56}{65}$ **(1)**

$\therefore \sin(x+y) = \dfrac{56}{65} \Rightarrow x + y = \sin^{-1}\left(\dfrac{56}{65}\right)$

$\because \quad x = \cos^{-1}\left(\dfrac{12}{13}\right)$ and $y = \sin^{-1}\left(\dfrac{3}{5}\right)$

$\therefore \cos^{-1}\left(\dfrac{12}{13}\right) + \sin^{-1}\left(\dfrac{3}{5}\right) = \sin^{-1}\left(\dfrac{56}{65}\right)$ **(1)**

Hence proved.

45. To prove

$$\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x}{1-3x^2}\right)$$

$$\text{LHS} = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$= \tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right] \quad \textbf{(1½)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right), \text{ if } \frac{2x^2}{1-x^2}<1 \quad \textbf{(1½)}$$

$$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \text{ if } 3x^2<1$$

$$\text{or if } x^2<\frac{1}{3} \text{ or } |x|<\frac{1}{\sqrt{3}} \quad \textbf{(1)}$$

$= \text{RHS}$

$\therefore$ LHS $=$ RHS **Hence proved.**

Alternate Method

Let $\tan^{-1}x=\theta$

Then, $x=\tan\theta$ **(1/2)**

$$\text{RHS} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$[\because x=\tan\theta]$ **(1½)**

$$= \tan^{-1}(\tan 3\theta)$$

$$\left[\because \tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= 3\theta = 3\tan^{-1}x \quad [\because \theta=\tan^{-1}x] \quad \textbf{(1)}$$

$$= \tan^{-1}x + 2\tan^{-1}x$$

$$= \tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \textbf{(1)}$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$= \text{LHS}$

$\therefore$ RHS $=$ LHS **Hence proved.**

46. To prove

$$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$$

$$\text{LHS} = \cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]$$

Let $\cot^{-1}x=\theta$

$\Rightarrow$ $x=\cot\theta$ **(1/2)**

Then, given expression may be written as

$$\cos[\tan^{-1}(\sin\theta)] = \cos\left[\tan^{-1}\left(\frac{1}{\operatorname{cosec}\theta}\right)\right] \quad \textbf{(1/2)}$$

$$\left[\because \operatorname{cosec}\theta=\frac{1}{\sin\theta}\right]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+\cot^2\theta}}\right)\right]$$

$$[\because \operatorname{cosec}^2\theta = 1+\cot^2\theta]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right] \quad [\because \cot\theta=x]$$

$$= \cos\phi \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\phi \text{ or } \tan\phi=\frac{1}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{\sec\phi} \quad \left[\because \sec\theta=\frac{1}{\cos\theta}\right]$$

$$= \frac{1}{\sqrt{1+\tan^2\phi}} \quad [\because \tan^2\theta+1=\sec^2\theta]$$

$$= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} \quad \left[\because \tan\phi=\frac{1}{\sqrt{1+x^2}}\right] \quad \textbf{(1)}$$

$$= \frac{1}{\sqrt{\frac{1+x^2+1}{1+x^2}}} = \sqrt{\frac{1+x^2}{2+x^2}}$$

$= \text{RHS}$ **(1)**

$\therefore$ LHS $=$ RHS **Hence proved.**

47. We are given that,

$$\cos^{-1}x+\sin^{-1}\left(\frac{x}{2}\right)=\frac{\pi}{6}$$

$$\Rightarrow \cos^{-1}x = \frac{\pi}{6}-\sin^{-1}\frac{x}{2}$$

$\Rightarrow \quad x = \cos\left(\frac{\pi}{6} - \sin^{-1}\frac{x}{2}\right)$

$\Rightarrow \quad x = \cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right) + \sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right)$ **(1)**

$[\because \cos(x-y) = \cos x \cos y + \sin x \sin y]$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\cos\left(\sin^{-1}\frac{x}{2}\right) + \frac{1}{2}\cdot\frac{x}{2}$

$[\because \sin(\sin^{-1}\theta) = \theta]$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$

$\left[\because \sin^{-1}\frac{x}{2} = \cos^{-1}\left(\sqrt{1-\frac{x^2}{4}}\right)\right]$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$

$\Rightarrow \quad x - \frac{x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$

$\Rightarrow \quad \frac{3x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$ **(1)**

On squaring both sides, we get

$\frac{9x^2}{16} = \frac{3}{4}\left(1-\frac{x^2}{4}\right)$

$\Rightarrow \quad \frac{3}{4}x^2 = 1 - \frac{x^2}{4} \Rightarrow \frac{3}{4}x^2 + \frac{x^2}{4} = 1$

$\Rightarrow \quad \frac{4x^2}{4} = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ **(1)**

But $x = -1$, do not satisfy the given equation.
Hence, $x = 1$ **(1)**

NOTE While solving an equation, please be careful on squaring the equation. Sometimes it may occurs extra value, which do not satisfy the given equation.

48. To prove $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

$\text{LHS} = 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$ ...(i)

We know that, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Using this identity, we can write

$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left[\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right] = \tan^{-1}\left(\frac{2/3}{1-\frac{1}{9}}\right)$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$ **(1½)**

On putting the value of $2\tan^{-1}\left(\frac{1}{3}\right)$ in Eq. (i), we get

$\text{LHS} = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$ **(1½)**

$= \tan^{-1}(1)$

$\because$ The principal value of $\tan^{-1}x$ is $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$\therefore \text{LHS} = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS}$ **(1)**

$\therefore \quad \text{LHS} = \text{RHS}$

Hence proved.

49. We are given that,

$\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}, \sqrt{6} > x > 0$

Using the identity in given equation, we get

$\tan^{-1}\left(\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x^2}{6}}\right) = \frac{\pi}{4}$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$ **(1½)**

$$\Rightarrow \frac{\frac{3x+2x}{6}}{\frac{6-x^2}{6}} = \tan\frac{\pi}{4} \Rightarrow \frac{5x}{6-x^2} = 1 \quad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\Rightarrow \quad 5x = 6 - x^2$$

$$\Rightarrow \quad x^2 + 5x - 6 = 0$$

$$\Rightarrow \quad x^2 + 6x - x - 6 = 0$$

$$\Rightarrow \quad x(x+6) - 1(x+6) = 0$$

$$\Rightarrow \quad (x-1)(x+6) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } -6 \qquad \textbf{(1½)}$$

But given that, $\sqrt{6} > x > 0 \Rightarrow x > 0$

$\therefore x = -6$ is rejected. **(1)**

Hence, $x = 1$

50. Apply $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ in LHS of given equation and then proceed further to obtain the desired result.

Given equation is

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right), x > 0$$

On applying identity in given equation, we get

$$\tan^{-1}\left[\frac{(x+2)+(x-2)}{1-(x+2)(x-2)}\right] = \tan^{-1}\left(\frac{8}{79}\right) \qquad \textbf{(1½)}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \quad \left[\frac{2x}{1-(x^2-4)}\right] = \frac{8}{79} \qquad \textbf{(1/2)}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \quad \frac{2x}{5-x^2} = \frac{8}{79} \Rightarrow \frac{x}{5-x^2} = \frac{4}{79}$$

$$\Rightarrow \quad 79x = 20 - 4x^2$$

$$\Rightarrow \quad 4x^2 + 79x - 20 = 0$$

$$\Rightarrow \quad 4x^2 + 80x - x - 20 = 0$$

$$\Rightarrow \quad 4x(x+20) - 1(x+20) = 0$$

$$\Rightarrow \quad (4x-1)(x+20) = 0$$

$$\Rightarrow \quad x = \frac{1}{4} \text{ or } -20 \qquad \textbf{(1)}$$

But given that, $x > 0$

$\therefore \quad x = -20$ is rejected.

Hence, $x = \frac{1}{4}$ **(1)**

51. Given equation is

$$2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] \qquad \textbf{(1½)}$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \quad [\because \sin^2 x = 1 - \cos^2 x] \ldots\text{(i)}$$

$$\Rightarrow \quad \sin x \cdot \cos x - \sin^2 x = 0$$

$$\Rightarrow \quad \sin x(\cos x - \sin x) = 0$$

$$\Rightarrow \quad \sin x = 0$$

$$\text{or} \quad \cos x = \sin x$$

$$\Rightarrow \quad \sin x = \sin 0$$

$$\text{or} \quad \cot x = 1 = \cot\frac{\pi}{4}$$

$$\Rightarrow \quad x = 0 \text{ or } \frac{\pi}{4} \qquad \textbf{(1½)}$$

But here at $x = 0$, the given equation does not exist at Eq. (i).

Hence, $x = \frac{\pi}{4}$ is the only solution. **(1)**

52. To prove

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$$

$$\text{LHS} = \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$[\because \sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})]$$

(1)

$$= \sin^{-1}\left(\frac{4}{5} \times \sqrt{\frac{144}{169}} + \frac{5}{13} \times \sqrt{\frac{9}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left[\left(\frac{4}{5} \times \frac{12}{13}\right) + \left(\frac{5}{13} \times \frac{3}{5}\right)\right] + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{48}{65} + \frac{15}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$= \sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$ **(1)**

$= \sin^{-1}\left[\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$

$[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})]$

$= \sin^{-1}\left(\frac{63}{65}\sqrt{\frac{4225-256}{4225}} + \frac{16}{65}\sqrt{\frac{4225-3969}{4225}}\right)$

$= \sin^{-1}\left(\frac{63}{65}\times\sqrt{\frac{3969}{4225}} + \frac{16}{65}\times\sqrt{\frac{256}{4225}}\right)$

$= \sin^{-1}\left(\frac{63}{65}\times\frac{63}{65} + \frac{16}{65}\times\frac{16}{65}\right)$

$= \sin^{-1}\left(\frac{3969+256}{4225}\right) = \sin^{-1}\left(\frac{4225}{4225}\right)$

$= \sin^{-1}(1)$ **(1)**

$\because$ The principal value of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore$ LHS $= \sin^{-1}\left(\sin\frac{\pi}{2}\right) = \frac{\pi}{2}$

$=$ RHS **(1)**

$\therefore$ LHS = RHS **Hence proved.**

Alternate Method

Given equation,

$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$ ...(i)

$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{16}{65}\right)$

$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{16}{65}\right)$ ...(ii)

$\left(\because \frac{\pi}{2} - \sin^{-1}\theta = \cos^{-1}\theta\right)$ **(1)**

Hence, Eqs. (i) and (ii) are equivalent. Now, we have to prove Eq. (ii).

Let $x = \sin^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \sin x = \frac{4}{5}$

and $y = \sin^{-1}\left(\frac{5}{13}\right) \Rightarrow \sin y = \frac{5}{13}$ **(1)**

Now, using the identity,

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta}$

$\therefore \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{16}{25}}$

$= \sqrt{\frac{9}{25}} = \frac{3}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{25}{169}}$

$= \sqrt{\frac{144}{169}} = \frac{12}{13}$ **(1)**

Using the identity,

$\cos(x+y) = \cos x\cdot\cos y - \sin x\cdot\sin y$

$= \frac{3}{5}\times\frac{12}{13} - \frac{4}{5}\times\frac{5}{13}$

$= \frac{36}{65} - \frac{20}{65} = \frac{16}{65}$

$\therefore x + y = \cos^{-1}\left(\frac{16}{65}\right)$

$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{16}{65}\right)$ **(1)**

$\left[\because x = \sin^{-1}\left(\frac{4}{5}\right) \text{ and } y = \sin^{-1}\left(\frac{5}{13}\right)\right]$

Hence proved.

53. To prove $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$

LHS $= \left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right) - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{9}{20}}\right) - \tan^{-1}\frac{8}{19}$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$ **(1)**

$= \tan^{-1}\left(\frac{27/20}{11/20}\right) - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right)$ **(1/2)**

$$= \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right) \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right)$$

$$= \tan^{-1}\left(\frac{425}{209}\times\frac{209}{425}\right) \quad \textbf{(1/2)}$$

$$= \tan^{-1}(1)$$

$\because$ The principal value of $\tan^{-1}x$ is $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

$$\therefore \quad \text{LHS} = \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} \quad \textbf{(1)}$$

$$= \text{RHS}$$

$\because$ LHS = RHS

Hence proved.

54. Applying the identity, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ in first two terms and the last two terms of LHS and then apply the same identity again to the get the RHS.

To prove $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

$$\text{LHS} = \left[\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)\right] + \left[\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]$$

On applying the result

$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$, we get

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\right)+\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right) \quad \textbf{(1½)}$$

$$= \tan^{-1}\left(\frac{8/15}{14/15}\right)+\tan^{-1}\left(\frac{15/56}{55/56}\right)$$

$$= \tan^{-1}\left(\frac{4}{7}\right)+\tan^{-1}\left(\frac{3}{11}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}\right) = \tan^{-1}\left(\frac{\frac{44+21}{77}}{\frac{77-12}{77}}\right) \quad \textbf{(1)}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left[\frac{65/77}{65/77}\right] \quad \textbf{(1)}$$

$$= \tan^{-1}(1)$$

$\because$ The principal value of $\tan^{-1}$ is $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

$$\therefore \quad \text{LHS} = \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} = \text{RHS} \quad \textbf{(1/2)}$$

$\because$ LHS = RHS

Hence proved.

55. The given equation is

$$\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$$

We know that,

$$\cot^{-1}x = \tan^{-1}\frac{1}{x}$$

So, the given equation can be written as

$$\tan^{-1}x + 2\tan^{-1}\left(\frac{1}{x}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x + \tan^{-1}\left(\frac{2\times\frac{1}{x}}{1-\frac{1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\left[\because 2\tan^{-1}\theta = \tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right)\right]$$

$$\Rightarrow \quad \tan^{-1}x + \tan^{-1}\left(\frac{\frac{2}{x}}{\frac{x^2-1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1} x + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3} \quad \textbf{(1½)}$$

$$\Rightarrow \tan^{-1}\left(\frac{x + \frac{2x}{x^2-1}}{1 - \frac{2x^2}{x^2-1}}\right) = \frac{2\pi}{3}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \frac{x^3 - x + 2x}{x^2 - 1 - 2x^2} = \tan\frac{2\pi}{3} \quad \textbf{(1½)}$$

$$\Rightarrow \frac{x^3 + x}{-1 - x^2} = \tan\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \frac{x^3 + x}{-(1+x^2)} = -\tan\frac{\pi}{3}$$

$$[\because \tan(\pi - \theta) = -\tan\theta]$$

$$\Rightarrow \frac{x(1+x^2)}{-(1+x^2)} = -\sqrt{3}$$

$$\Rightarrow x = \sqrt{3} \quad \textbf{(1)}$$

56. The given equation is

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{(x-1)(x+1)}{(x-2)(x+2)}}\right] = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \quad \textbf{(1½)}$$

$$\Rightarrow \frac{\frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2) - (x-1)(x+1)}{(x-2)(x+2)}} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x^2 + x - 2 + x^2 - x - 2}{(x^2-4) - (x^2-1)} = 1$$

$$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } (a-b)(a+b) = a^2 - b^2\right]$$

$$\Rightarrow \frac{2x^2 - 4}{-3} = 1 \quad \textbf{(1½)}$$

$$\Rightarrow 2x^2 - 4 = -3$$

$$\Rightarrow 2x^2 = -3 + 4 = 1$$

$$\Rightarrow 2x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}} \quad \textbf{(1)}$$

57. The given equation is

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$$

$$\Rightarrow \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4} \quad \textbf{(1/2)}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{1+x}{1-x} - x}{1 + \left(\frac{1+x}{1-x}\right)\cdot x}\right] = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right] \textbf{(1½)}$$

$$\Rightarrow \frac{1 + x - x + x^2}{1 - x + x + x^2} = \tan\frac{\pi}{4} \quad \textbf{(1)}$$

$$\Rightarrow \frac{1+x^2}{1+x^2} = 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\therefore 1 = 1$$

So, the given equation has many solutions. **(1)**